

FRACTIONAL SMOOTHNESS OF IMAGES OF LOGARITHMICALLY CONCAVE MEASURES UNDER POLYNOMIALS

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ABSTRACT. We show that a measure on the real line that is the image of a log-concave measure under a polynomial of degree d possesses a density from the Nikol'skii–Besov class of fractional order $1/d$. This result is used to prove an estimate of the total variation distance between such measures in terms of the Fortet–Mourier distance.

Keywords: Logarithmically concave measure, Total variation distance, Fortet–Mourier distance, Distribution of a polynomial, Nikol'skii–Besov class

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INTRODUCTION

Many fundamental problems of stochastic calculus involve investigation of the smoothness properties of measures of the form $\nu = \mu \circ f^{-1}$, i.e., measures induced by μ -measurable functions f with respect to a given measure μ on an infinite-dimensional space (e.g., the distribution of a stochastic process). In this paper we study the class of such measures ν induced by polynomials on spaces with logarithmically concave measures. Since all Gaussian measures are logarithmically concave, our results apply to Gaussian measures (e.g., to the Wiener measure). This class of distributions ν is of interest for many applications, because it contains typical statistics and the class of all polynomials of a fixed degree can be considered as an important family of nonlinear transformations of a given measure. Various properties of measures in the class under consideration have been studied in many works, see [11, 12, 13, 17, 24, 25, 27, 28] for the case of Gaussian measures and [1, 6, 7, 15, 22, 26] for the case of general logarithmically concave measures.

Our first main result states that the density of a polynomial image of a log-concave measure always belongs to the Nikol'skii–Besov class $B_{1,\infty}^{1/d}$ (see [4, 23], sometimes it is also denoted by $\Lambda_{1/d}^{1,\infty}$, see [29]), where d is the degree of the polynomial. We also prove the following quantitative estimate (Corollary 5.2):

$$\sigma_f^{1/d} \int_{\mathbb{R}} |\rho(t+h) - \rho(t)| dx \leq C(d) |h|^{1/d} \quad \forall h \in \mathbb{R}.$$

Here ρ is the density of the measure $\mu \circ f^{-1}$, μ is a log-concave measure, f is a polynomial of degree d , and σ_f^2 is the variance of f .

This result is used to obtain an estimate of the total variation distance between distributions of polynomials in terms of the Fortet–Mourier distance (Corollary 5.4):

$$\|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{TV}} \leq C(d, a) \|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{FM}}^{1/(1+d)},$$

provided that $\sigma_f, \sigma_g \geq a$. This estimate generalizes some recent results from [25, 26] and [11] to the case of log-concave measures. However, even in the case of a Gaussian measure the power at the Fortet–Mourier distance in our estimate is better in comparison with similar results from the cited papers.

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The paper is organized in the following way. In Section 1 we give necessary definitions and some preliminary results needed in the proofs of the main results. The subsequent four sections contain the proofs of our results. An important tool in our approach is the so-called localization technique (Theorem 1.5) that enables us to reduce certain high-dimensional inequalities to inequalities in low dimensions. This means that if we want to obtain a dimension-free estimate for the class of log-concave measures, we can prove a low-dimensional estimate and then use the localization techniques to make it dimension-free. Let us outline some key steps in each section. The main tool of studying smoothness of induced distributions is the Malliavin method, but in our case it cannot be applied directly, since the density of a polynomial distribution need not be even bounded (e.g., take the χ^2 -distribution with one degree of freedom). To overcome this difficulty in Section 2 we obtain a sufficient Malliavin-type condition for the density of a measure on the real line to belong to the Nikol'skii–Besov class (Lemma 2.1) and from this we deduced an estimate of the total variation distance in terms of the Fortet–Mourier distance (Lemma 2.3). An important ingredient of the classical Malliavin method is a certain smoothness requirement on the measure μ the images of which we study. Since our approach does not allow to deal with each fixed log-concave measure, but instead deals with the whole class of log-concave measures, we have to provide an estimate on the derivatives of a log-concave measure that does not depend much on the measure. We obtain such an estimate in Section 3, where we prove an estimate on the variation of the Skorohod derivative of an isotropic log-concave measure on \mathbb{R}^n in terms of its isotropic constant. In Section 4 we combine the results of the previous section with the localization techniques to verify our Malliavin-type condition from Section 2 for the polynomial images of log-concave measures (Theorem 4.4). Note that another important ingredient of the classical Malliavin method is a certain nondegeneracy condition imposed on the mapping f that induces the distribution under consideration. In our case when f is a polynomial, we automatically have such a nondegeneracy condition in the form of the Carbery–Wright inequality (Theorem 1.3). Finally, in Section 5 we present our main results (Corollaries 5.2, 5.3, 5.4, and 5.5) for log-concave measures on infinite dimensional locally convex spaces that follow from the technical result of Theorem 4.4 and an approximation argument.

1. PRELIMINARIES

In this section we introduce necessary definitions and notation. We also formulate here some auxiliary results.

For $x, y \in \mathbb{R}^n$ let (x, y) denote the standard Euclidian inner product in \mathbb{R}^n and let $|x|$ be the norm generated by this inner product, i.e. $|x| := \sqrt{(x, x)}$. Let $C_0^\infty(\mathbb{R}^n)$ denote the space of all smooth functions with compact support and let $C_b^\infty(\mathbb{R}^n)$ denote the space of all bounded smooth functions with bounded derivatives of all orders. For a function φ on the real line we set

$$\|\varphi\|_\infty := \sup_t |\varphi(t)|.$$

The total variation and the Fortet–Mourier distances between two probability measures ν_1 and ν_2 on \mathbb{R} are defined by the following equalities, respectively:

$$\|\nu_1 - \nu_2\|_{\text{TV}} := \sup \left\{ \int \varphi d(\nu_1 - \nu_2), \varphi \in C_b^\infty(\mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\},$$

$$\|\nu_1 - \nu_2\|_{\text{FM}} := \sup \left\{ \int \varphi d(\nu_1 - \nu_2), \varphi \in C_b^\infty(\mathbb{R}^n), \|\varphi\|_\infty \leq 1, \|\varphi'\|_\infty \leq 1 \right\}.$$

Note that $\|\nu_1 - \nu_2\|_{\text{FM}} \leq 2$.

A probability Borel measure μ on \mathbb{R}^n is called logarithmically concave (log-concave or convex) if it has a density of the form e^{-V} with respect to Lebesgue measure on some affine subspace L ,

where $V: L \rightarrow (-\infty, +\infty]$ is a convex function. This definition is equivalent to the property that for every pair of Borel sets A, B the following inequality holds (see [14]):

$$\mu(tA + (1-t)B) \geq \mu(A)^t \mu(B)^{1-t} \quad \forall t \in [0, 1].$$

A Radon probability measure μ on a locally convex space E is called log-concave (or convex) if its images under continuous linear operators to \mathbb{R}^n are log-concave.

Let K be a convex body in \mathbb{R}^n with $0 \in K$. Its Minkowski functional $\|\cdot\|_K$ is defined by

$$\|x\|_K := \inf\{t > 0: t^{-1}x \in K\}.$$

By convex bodies we mean closed convex sets with non empty interior.

Let I_A denote the indicator function of the set A and let $|A|$ denote the Lebesgue volume of the set A . The symbol λ denotes the standard Lebesgue measure on the real line.

A log-concave measure μ on \mathbb{R}^n is called isotropic if it is absolutely continuous with respect to Lebesgue measure and

$$\int_{\mathbb{R}^n} (x, \theta) \mu(dx) = 0, \quad \int_{\mathbb{R}^n} (x, \theta)^2 \mu(dx) = L_\mu^2 |\theta|^2 \quad \forall \theta \in \mathbb{R}^n,$$

where the constant L_μ is called the isotropic constant of the measure μ .

Let μ be a Borel probability measure on \mathbb{R}^n and let $h \in \mathbb{R}^n$. The Skorohod derivative $D_h \mu$ of the measure μ along h is a bounded signed Borel measure on \mathbb{R}^n such that

$$\int_X \partial_h \varphi(x) \mu(dx) = - \int_X \varphi(x) D_h \mu(dx)$$

for every $\varphi \in C_b^\infty(\mathbb{R}^n)$ (see [10]). A measure μ has the Skorohod derivatives along all vectors precisely when it possesses a density of class BV (the class of functions of bounded variation).

According to Krugova's result [20] (see also [10, Section 4.3]), for every log-concave measure μ with a density ρ and for every vector h of unit length one has the following equality:

$$\|D_h \mu\|_{TV} = 2 \int_{\langle h \rangle^\perp} \max_t \rho(x + th) dx,$$

where $\langle h \rangle^\perp$ is the orthogonal complement of h .

Let ν be a Borel probability measure on the real line and let ν_h denote its shift by the vector h :

$$\nu_h(A) := \nu(A - h).$$

Let μ be a Radon probability measure on a locally convex space E . Denote by $\mathcal{P}^d(\mu)$ the closure in $L^2(\mu)$ of the set of functions of the form $f(\ell_1, \dots, \ell_n)$, where $\ell_i \in E^*$ (the topological dual space to E) and f is a polynomial on \mathbb{R}^n of degree d . It is shown in [2] that every function from $\mathcal{P}^d(\mu)$ has a version that is a polynomial of degree d in the usual algebraic sense.

For a μ -measurable function f , let

$$\|f\|_r = \left(\int |f|^r d\mu \right)^{1/r} \quad \text{for } r > 0, \quad \|f\|_0 = \exp \left(\int \ln |f| d\mu \right) = \lim_{r \rightarrow 0} \|f\|_r,$$

$$\mathbb{E}f := \int f d\mu \text{ is the expectation of the random variable } f,$$

$$\sigma_f^2 = \int (f - \mathbb{E}f)^2 d\mu \text{ is the variance of the random variable } f.$$

We recall (see [4, 23, 29]) that a function $\rho \in L^1(\mathbb{R})$ belongs to the Nikol'skii–Besov class $B_{1,\infty}^\alpha$ ($0 < \alpha < 1$) if there is a constant $C > 0$ such that

$$\int |\rho(x+h) - \rho(x)| dx \leq C|h|^\alpha, \quad \forall h \in \mathbb{R}.$$

The following known results will be used in the proofs.

Theorem 1.1 (see [19, 3]). *For every $n \in \mathbb{N}$, there is a constant C_n depending only on n such that for every isotropic log-concave measure μ on \mathbb{R}^n with a density ρ and the isotropic constant L_μ one has*

$$(\max \rho)^{1/n} L_\mu \leq C_n.$$

There is an open conjecture (the hyperplane conjecture) that the constant in the previous theorem can be chosen independent of n , but for now the best known constant $C_n \sim n^{1/4}$ is due to Klartag (see [19]).

Theorem 1.2 (see [6, 7]). *There is an absolute constant c such that for every log-concave measure μ on \mathbb{R}^n , for every number $q \geq 1$ and for every polynomial f of degree d on \mathbb{R}^n the following inequality holds:*

$$\|f\|_q \leq (cqd)^d \|f\|_p, \text{ whenever } 0 \leq p < q < \infty.$$

The following estimate is usually called the Carbery–Wright inequality (see [15], it was also implicitly proved in [22]).

Theorem 1.3 (see [15, 22]). *There is an absolute constant c_1 such that for every log-concave measure μ on \mathbb{R}^n and for every polynomial f of degree d the following inequality holds:*

$$\mu(|f| \leq t) \left(\int |f| d\mu \right)^{1/d} \leq t^{1/d} c_1 d.$$

Some analogues of the previous two theorems for measurable polynomials on infinite-dimensional spaces are discussed in [1].

The following assertion is the Poincaré inequality for log-concave measures.

Theorem 1.4 (see [5, 18]). *There is an absolute constant R such that for every log-concave measure μ on \mathbb{R}^n and for every locally Lipschitz function f on \mathbb{R}^n one has*

$$\int (f - \mathbb{E}f)^2 d\mu \leq R \int |x - x_0|^2 d\mu \int |\nabla f|^2 d\mu, \text{ where } x_0 = \int x d\mu.$$

We also need the following result of Fradelizi and Guédon that generalizes the localization lemma from [21, 18].

Theorem 1.5 (Localization lemma with p constraints, see [16]). *Let K be a compact convex set in \mathbb{R}^n , $f_i: K \rightarrow \mathbb{R}$, $1 \leq i \leq p$. Assume that all functions f_i are upper semi-continuous. Let P_{f_1, \dots, f_p} be the set of all log-concave measures with support in K such that*

$$\int f_i d\mu \geq 0, \quad i = 1, \dots, p.$$

Let $\Phi: P(K) \rightarrow \mathbb{R}$ be a convex upper semi-continuous function, where $P(K)$ is the space of all Borel probability measures on K equipped with the weak topology. Then $\sup_{\mu \in P_{f_1, \dots, f_p}} \Phi(\mu)$ is attained on log-concave measures μ such that the smallest affine subspace containing the support of μ is of dimension not greater than p .

2. SUFFICIENT CONDITIONS FOR FRACTIONAL SMOOTHNESS OF MEASURES ON THE REAL LINE

In this section we provide a Malliavin-type condition for the density of a measure on the real line to belong to the Nikol'skii–Besov class. We also prove several estimates for different distances between measures satisfying this condition. In this section we assume that the parameter α belongs to $(0, 1]$.

The proofs of Lemma 2.1 and Lemma 2.3 below can be found in [11].

Lemma 2.1. *Let ν be a probability Borel measure on the real line. Assume that for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has*

$$\int \varphi' d\nu \leq C \|\varphi'\|_\infty^{1-\alpha}.$$

Then

$$\|\nu_h - \nu\|_{\text{TV}} \leq 2^{1-\alpha} C |h|^\alpha \quad \forall h \in \mathbb{R}.$$

Remark 2.2. Note that the condition

$$\|\nu_h - \nu\|_{\text{TV}} \leq C |h|^\alpha$$

is equivalent to the property that the measure ν possesses a density from the Nikol'skii–Besov class $B_{1,\infty}^\alpha$.

Lemma 2.3. *Let ν, σ be a pair of probability Borel measures on the real line such that*

$$\|\nu_h - \nu\|_{\text{TV}} \leq C_\nu |h|^\alpha, \quad \|\sigma_h - \sigma\|_{\text{TV}} \leq C_\sigma |h|^\alpha$$

for some number $\alpha > 0$. Then

$$\|\sigma - \nu\|_{\text{TV}} \leq C(\nu, \sigma) \|\sigma - \nu\|_{\text{FM}}^{\frac{\alpha}{1+\alpha}},$$

where

$$C(\nu, \sigma) = 2 + (C_\sigma + C_\nu)(2\pi)^{-1/2} \int e^{-\frac{t^2}{2}} |t|^\alpha dt.$$

Lemma 2.4. *Let ν be a Borel probability measure on the real line. Assume that for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has*

$$\int \varphi' d\nu \leq C \|\varphi'\|_\infty^{1-\alpha}.$$

Then, for every Borel set A , the following estimate holds:

$$\nu(A) \leq C \lambda(A)^\alpha,$$

where λ is the standard Lebesgue measure on the real line. Moreover, if ρ_ν is the density of ν , then $\rho_\nu \in L^p(\lambda)$ whenever $1 < p < \frac{1}{1-\alpha}$ and one has

$$\|\rho_\nu\|_{L^p(\lambda)} \leq \left(p(p-1)^{-1} + p \left(\frac{1}{1-\alpha} - p \right)^{-1} \right)^{1/p} C^{\frac{1}{\alpha}(1-1/p)}.$$

Proof. For any $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ we have

$$\int \varphi d\nu = \int \Phi' d\nu,$$

where

$$\Phi(s) := \int_{-\infty}^s \varphi(t) dt.$$

Note that

$$|\Phi(s)| \leq \int |\varphi(t)| dt.$$

By the assumptions of the lemma

$$\left(\int |\varphi(t)| dt \right)^{-1} \int \Phi'(s) \nu(ds) \leq C \left(\int |\varphi(t)| dt \right)^{-1+\alpha},$$

so

$$\int \varphi d\nu \leq C \left(\int |\varphi(t)| dt \right)^\alpha.$$

Now the first estimate of the lemma can be obtained by approximation. Let us prove the second part. Note that

$$t\lambda(\rho_\nu \geq t) \leq \int_{\{\rho_\nu \geq t\}} \rho_\nu d\lambda \leq 1.$$

Applying the first part of the lemma we get

$$t\lambda(\rho_\nu \geq t) \leq \int_{\{\rho_\nu \geq t\}} \rho_\nu d\lambda = \nu(\rho_\nu \geq t) \leq C\lambda(\rho_\nu \geq t)^\alpha,$$

therefore,

$$\lambda(\rho_\nu \geq t) \leq t^{-\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha}}.$$

Using these estimates and the Fubini theorem, we obtain

$$\begin{aligned} \int \rho_\nu^p d\lambda &= p \int_0^\infty t^{p-1} \lambda(\rho_\nu \geq t) dt = p \left(\int_0^\tau + \int_\tau^\infty \right) t^{p-1} \lambda(\rho_\nu \geq t) dt \\ &\leq p \int_0^\tau t^{p-2} dt + C^{\frac{1}{1-\alpha}} p \int_\tau^\infty t^{p-1-\frac{1}{1-\alpha}} dt = p(p-1)^{-1} \tau^{p-1} + C^{\frac{1}{1-\alpha}} p \left(\frac{1}{1-\alpha} - p \right)^{-1} \tau^{p-\frac{1}{1-\alpha}}. \end{aligned}$$

Taking now $\tau = C^{1/\alpha}$, we get

$$\int \rho_\nu^p d\lambda \leq \left(p(p-1)^{-1} + p \left(\frac{1}{1-\alpha} - p \right)^{-1} \right) C^{(p-1)/\alpha}.$$

The lemma is proved. \square

Remark 2.5. Similarly to the previous lemma one can prove that for a function $f \in L^1(\lambda)$ such that for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$

$$\int \varphi' f d\lambda \leq C \|\varphi'\|_\infty^{1-\alpha}$$

one has

$$\|f\|_{L^p(\lambda)} \leq \left(p(p-1)^{-1} + p \left(\frac{1}{1-\alpha} - p \right)^{-1} \right)^{1/p} \|f\|_{L^1(\lambda)}^{1-\frac{1}{\alpha}(1-1/p)} C^{\frac{1}{\alpha}(1-1/p)}$$

whenever $1 < p < \frac{1}{1-\alpha}$. Thus, if ν, σ is a pair of Borel probability measures on the real line such that for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has

$$\int \varphi' d\nu \leq C_\nu \|\varphi'\|_\infty^{1-\alpha}, \quad \int \varphi' d\sigma \leq C_\sigma \|\varphi'\|_\infty^{1-\alpha}$$

and ρ_ν, ρ_σ are their densities, then

$$\|\rho_\nu - \rho_\sigma\|_{L^p(\lambda)} \leq \left(p(p-1)^{-1} + p \left(\frac{1}{1-\alpha} - p \right)^{-1} \right)^{1/p} \|\sigma - \nu\|_{\text{TV}}^{1-\frac{1}{\alpha}(1-1/p)} (C_\nu + C_\sigma)^{\frac{1}{\alpha}(1-1/p)}$$

for $1 < p < \frac{1}{1-\alpha}$.

3. ESTIMATES FOR THE SKOROHOD DERIVATIVES OF ISOTROPIC LOG-CONCAVE MEASURES

As it has already been said in the introduction, here we obtain an estimate on the variation of the Skorohod derivative of an isotropic log-concave measure in terms of its isotropic constant. The idea is quite simple. If a log-concave measure can be viewed as a generalization of the uniform distribution on a convex compact set, then the norm of the Skorohod derivative of this measure can be viewed as a generalization of the volume of the projection of the convex set. To estimate the volume of the projection we can embed our convex set into some ball. Thus, to estimate the Skorohod derivative of the isotropic log-concave measure on \mathbb{R}^n with a density ρ and the unit isotropic constant we will find constants $c_n > 0$ and $\alpha_n > 0$ such that

$$\rho(x) \leq c_n e^{-\alpha_n |x|}.$$

Throughout this section we assume that $\tau \in (0, \infty)$.

Lemma 3.1. *Let μ be a log-concave measure on \mathbb{R}^n with a density ρ . Let $m_\rho = \max \rho$ and let*

$$K = \{x \in \mathbb{R}^n : \rho(x) \geq e^{-\tau} m_\rho\}.$$

Then for the volume of the body K the following estimate holds true:

$$1 \leq m_\rho c_n(\tau) |K|,$$

$$\text{where } c_n(\tau) = n \int_1^\infty t^{n-1} e^{-\tau t} dt + 1.$$

Proof. Without loss of generality we can assume that $m_\rho = \rho(0)$. Since μ is a probability measure, we have

$$1 = \int \rho(x) dx = \int_{\mathbb{R}^n \setminus K} \rho(x) dx + \int_K \rho(x) dx.$$

By using polar coordinates we can estimate the first integral in this sum as follows:

$$\begin{aligned} \int_{\mathbb{R}^n \setminus K} \rho(x) dx &= \int_{S^{n-1}} \int_{\|\varphi\|_K^{-1}}^\infty r^{n-1} \rho(r\varphi) dr \sigma_{n-1}(d\varphi) \\ &= \int_{S^{n-1}} \|\varphi\|_K^{-n} \int_1^\infty t^{n-1} \rho(t\|\varphi\|_K^{-1}\varphi) dt \sigma_{n-1}(d\varphi). \end{aligned}$$

Since $\rho(x) = e^{-V(x)}$ with a convex function V , we have

$$V(t\|\varphi\|_K^{-1}\varphi) \geq tV(\|\varphi\|_K^{-1}\varphi) + (1-t)V(0)$$

whenever $t \geq 1$. So,

$$\rho(t\|\varphi\|_K^{-1}\varphi) \leq \rho(0) \left(\frac{\rho(\|\varphi\|_K^{-1}\varphi)}{\rho(0)} \right)^t = \rho(0) e^{-\tau t}.$$

Thus,

$$\begin{aligned} \int_{S^{n-1}} \|\varphi\|_K^{-n} \int_1^\infty t^{n-1} \rho(t\|\varphi\|_K^{-1}\varphi) dt \sigma_{n-1}(d\varphi) &\leq \rho(0) \int_{S^{n-1}} \|\varphi\|_K^{-n} \int_1^\infty t^{n-1} e^{-\tau t} dt \sigma_{n-1}(d\varphi) \\ &= \rho(0) (c_n(\tau) - 1) n^{-1} \int_{S^{n-1}} \|\varphi\|_K^{-n} \sigma_{n-1}(d\varphi) \\ &= \rho(0) (c_n(\tau) - 1) \int_{S^{n-1}} \int_0^{\|\varphi\|_K^{-1}} r^{n-1} dr \sigma_{n-1}(d\varphi) = \rho(0) (c_n(\tau) - 1) |K|. \end{aligned}$$

Hence, $1 \leq \rho(0) c_n(\tau) |K|$, and the lemma is proved. \square

The proof of the following lemma is a combination of the proof of Theorem 4.1 from [18] and the previous lemma.

Lemma 3.2. *Let μ be an isotropic log-concave measure on \mathbb{R}^n with a density ρ and the isotropic constant 1. Let $m_\rho = \max \rho$ and let*

$$K = \{x \in \mathbb{R}^n : \rho(x) \geq e^{-\tau} m_\rho\}.$$

Then for every point $x \in K$ we have

$$|x|^2 \leq c_n(\tau) (n+1)^2 e^\tau,$$

$$\text{where } c_n(\tau) = n \int_1^\infty t^{n-1} e^{-\tau t} dt + 1.$$

Proof. Let $v \in K$. Note that

$$\int_K (x, \theta)^2 dx = \int_{K-v} (v + u, \theta)^2 du = \int_{S^{n-1}} \int_0^{\|\varphi\|_{K-v}^{-1}} r^{n-1} (v + r\varphi, \theta)^2 dr \sigma_{n-1}(d\varphi),$$

where in the second equality we pass to polar coordinates. Calculating the inner integral in r we have

$$\begin{aligned} & \int_{S^{n-1}} (n^{-1} \|\varphi\|_{K-v}^{-n} (v, \theta)^2 + 2(n+1)^{-1} \|\varphi\|_{K-v}^{-n-1} (v, \theta) (\varphi, \theta) + (n+2)^{-1} \|\varphi\|_{K-v}^{-n-2} (\varphi, \theta)^2) \sigma_{n-1}(d\varphi) \\ &= \int_{S^{n-1}} \|\varphi\|_{K-v}^{-n} n^{-1} \left(\frac{\sqrt{n(n+2)}}{n+1} (v, \theta) + \sqrt{\frac{n}{n+2}} \|\varphi\|_{K-v}^{-1} (\varphi, \theta) \right)^2 \\ &+ n^{-1} (n+1)^{-2} \|\varphi\|_{K-v}^{-n} (v, \theta)^2 \sigma_{n-1}(d\varphi) \geq n^{-1} (n+1)^{-2} (v, \theta)^2 \int_{S^{n-1}} \|\varphi\|_{K-v}^{-n} \sigma_{n-1}(d\varphi), \end{aligned}$$

where the last inequality is valid, since the first term under the integral sign is nonnegative. In turn, the last expression is equal to

$$(n+1)^{-2} (v, \theta)^2 \int_{S^{n-1}} \int_0^{\|\varphi\|_{K-v}^{-1}} r^{n-1} dr \sigma_{n-1}(d\varphi) = (n+1)^{-2} (v, \theta)^2 |K|.$$

Thus, using the estimate for the volume of the body K from the previous lemma, we obtain

$$\begin{aligned} |\theta|^2 &= \int_K (x, \theta)^2 \rho(x) dx \geq e^{-\tau} m_\rho \int_K (x, \theta)^2 dx \\ &\geq e^{-\tau} m_\rho (n+1)^{-2} |K| (v, \theta)^2 \geq \frac{e^{-\tau}}{c_n(\tau) (n+1)^2} (v, \theta)^2. \end{aligned}$$

Now we can take $\theta = v$ and get the estimate $|v|^2 \leq c_n(\tau) (n+1)^2 e^\tau$. \square

Lemma 3.3. *For every $n \in \mathbb{N}$, there are constants $c_n > 0$ and $\alpha_n > 0$ depending only on n such that, for every isotropic log-concave measure μ on \mathbb{R}^n with a density ρ and the isotropic constant 1, the following inequality holds true:*

$$\rho(x) \leq c_n e^{-\alpha_n |x|}.$$

Proof. Let x_0 be a point such that $\rho(x_0) = \max \rho = m_\rho$,

$$K = \{x \in \mathbb{R}^n : \rho(x) \geq e^{-\tau} m_\rho\}, \quad r = (n+1) \sqrt{c_n(\tau) e^\tau},$$

where $c_n(\tau)$ is the constant from the previous lemma. By the previous lemma we have the inclusion $K \subset B_r$, where B_r is the ball of radius r centered at the origin.

Suppose first that $x \notin B_{2r}$, equivalently, $\|x - x_0\|_{B_{2r}-x_0} \geq 1$, where $\|\cdot\|_{B_{2r}-x_0}$ is the Minkowski functional of the set $B_{2r} - x_0$. Note that

$$x = (1 - \|x - x_0\|_{B_{2r}-x_0}) x_0 + \|x - x_0\|_{B_{2r}-x_0} \left(x_0 + \frac{x - x_0}{\|x - x_0\|_{B_{2r}-x_0}} \right).$$

Since μ is a log-concave measure, the density ρ is of the form e^{-V} with some convex function V . Using the equality mentioned above and the convexity of the function V we have

$$\begin{aligned} V(x) &\geq (1 - \|x - x_0\|_{B_{2r}-x_0}) V(x_0) + \|x - x_0\|_{B_{2r}-x_0} V \left(x_0 + \frac{x - x_0}{\|x - x_0\|_{B_{2r}-x_0}} \right) \\ &\geq (1 - \|x - x_0\|_{B_{2r}-x_0}) V(x_0) + \|x - x_0\|_{B_{2r}-x_0} (V(x_0) + \tau) = V(x_0) + \tau \|x - x_0\|_{B_{2r}-x_0}, \end{aligned}$$

where the inequality follows from the fact that the point $x_0 + \frac{x - x_0}{\|x - x_0\|_{B_{2r}-x_0}}$ does not belong to the set K .

We now estimate $\|x - x_0\|_{B_{2r-x_0}}$. Note that

$$\left| \frac{x - x_0}{\|x - x_0\|_{B_{2r-x_0}}} + x_0 \right| = 2r,$$

hence

$$2r\|x - x_0\|_{B_{2r-x_0}} \geq |x| - (\|x - x_0\|_{B_{2r-x_0}} - 1)|x_0| \geq |x| - (\|x - x_0\|_{B_{2r-x_0}} - 1)r,$$

which implies the estimate

$$\|x - x_0\|_{B_{2r-x_0}} \geq 1/3 + (3r)^{-1}|x|.$$

Thus, we have

$$V(x) \geq V(x_0) + \tau(1/3 + (3r)^{-1}|x|),$$

hence,

$$\rho(x) \leq e^{-\tau/3} m_\rho e^{-\tau(3r)^{-1}|x|} \leq m_\rho e^{-\tau(3r)^{-1}|x|} \leq e^\tau m_\rho e^{-\tau(3r)^{-1}|x|}.$$

We now consider the case $x \in B_{2r}$. In this case we have

$$\rho(x) \leq m_\rho \leq e^\tau m_\rho e^{-\tau(3r)^{-1}|x|}.$$

By Theorem 1.1 one has $m_\rho \leq C_n^n$ for some constant C_n , so, taking $\tau = 1$, we obtain the desired estimate. \square

Theorem 3.4. *Let μ be an isotropic log-concave measure on \mathbb{R}^n with a density ρ and the isotropic constant 1. Let h be a vector of unit length. Then*

$$\|D_h \mu\| \leq C(n)$$

with some constant $C(n)$ that depends only on the dimension n .

Proof. Without loss of generality we can assume that $h = e_1$ is the first basis vector. According to Krugova's result [20],

$$\frac{1}{2} \|D_h \mu\| = \int_{\mathbb{R}^{n-1}} \max_t \rho(t, x_2, \dots, x_n) dx_2 \dots dx_n.$$

Now we can use the estimate from Lemma 3.3:

$$\int_{\mathbb{R}^{n-1}} \max_t \rho(t, x_2, \dots, x_n) dx_2 \dots dx_n \leq c_n \int_{\mathbb{R}^{n-1}} \exp\left(-\alpha_n \left(\sum_{i=2}^n x_i^2\right)^{1/2}\right) dx_2 \dots dx_n,$$

where the last expression depends only on the dimension n , and the lemma is proved. \square

4. VERIFICATION OF THE FRACTIONAL SMOOTHNESS SUFFICIENT CONDITION

This section is devoted to obtaining a technical statement in Theorem 4.4. First, using estimates of the Skorohod derivatives from the previous section, the Poincaré inequality, and the Carbery–Wright inequality we obtain an estimate that depends on dimension. Then we use the localization techniques to make this estimate dimension-free.

Lemma 4.1. *Let μ be a log-concave measure on \mathbb{R}^n with a density ρ with respect to Lebesgue measure. Then, for every $d \in \mathbb{N}$, there are constants $c_1(d), c_2(d)$ depending only on d such that, for every polynomial f of degree d on \mathbb{R}^n , every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$, and for every vector e of unit length the following inequality holds true:*

$$\int \varphi'(f) d\mu \leq \left(c_1(d) \|\partial_e f\|_2^{-1/(d-1)} + c_2(d) \|D_e \mu\|_{\text{TV}} \right) \|\varphi'\|_\infty^{1-1/d}.$$

Proof. Fix a number $\varepsilon > 0$. Note that

$$\int \varphi'(f) d\mu = \int \frac{(\partial_e f)^2}{(\partial_e f)^2 + \varepsilon} \varphi'(f) d\mu + \varepsilon \int \varphi'(f) ((\partial_e f)^2 + \varepsilon)^{-1} d\mu.$$

Consider the first term:

$$\int \frac{(\partial_e f)^2}{(\partial_e f)^2 + \varepsilon} \varphi'(f) d\mu = \int \partial_e(\varphi(f)) \frac{\partial_e f}{(\partial_e f)^2 + \varepsilon} d\mu.$$

Applying the integration by parts formula to the last expression (recall that $D_e \mu$ denotes the Skorohod derivative of the measure μ along the vector e) we have

$$\begin{aligned} & - \int \varphi(f) \left[\frac{\partial_e^2 f}{(\partial_e f)^2 + \varepsilon} - 2 \frac{(\partial_e f)^2 \partial_e^2 f}{((\partial_e f)^2 + \varepsilon)^2} \right] d\mu - \int \varphi(f) \frac{\partial_e f}{(\partial_e f)^2 + \varepsilon} d(D_e \mu) \\ & \leq \int \left| \frac{\partial_e^2 f}{(\partial_e f)^2 + \varepsilon} \right| d\mu + 2 \int \left| \frac{(\partial_e f)^2 \partial_e^2 f}{((\partial_e f)^2 + \varepsilon)^2} \right| d\mu + \int \left| \frac{\partial_e f}{(\partial_e f)^2 + \varepsilon} \right| d|D_e \mu| \\ & = \varepsilon^{-1/2} \left(\int \left| \frac{\partial_e^2 g}{(\partial_e g)^2 + 1} \right| d\mu + 2 \int \left| \frac{(\partial_e g)^2 \partial_e^2 g}{((\partial_e g)^2 + 1)^2} \right| d\mu + \int \left| \frac{\partial_e g}{(\partial_e g)^2 + 1} \right| d|D_e \mu| \right), \end{aligned}$$

where $g = f\varepsilon^{-1/2}$. Now let us estimate each term separately:

$$\begin{aligned} & \int \left| \frac{\partial_e g}{(\partial_e g)^2 + 1} \right| d|D_e \mu| \leq 1/2 \|D_e \mu\|_{\text{TV}}, \\ & \int \left| \frac{\partial_e^2 g}{(\partial_e g)^2 + 1} \right| d\mu = \int_{\langle e \rangle^\perp} \int_{\langle e \rangle} \left| \frac{\partial_e^2 g(x + te)}{(\partial_e g(x + te))^2 + 1} \right| \rho(x + te) dt dx \\ & \leq d \int_{\langle e \rangle^\perp} \max_s \rho(x + se) \int_{\langle e \rangle} \left| \frac{1}{\tau^2 + 1} \right| d\tau dx = 1/2 d\pi \|D_e \mu\|_{\text{TV}}, \\ & \int \left| \frac{(\partial_e g)^2 \partial_e^2 g}{((\partial_e g)^2 + 1)^2} \right| d\mu \leq \int \left| \frac{\partial_e^2 g}{(\partial_e g)^2 + 1} \right| d\mu \leq 1/2 d\pi \|D_e \mu\|_{\text{TV}}. \end{aligned}$$

Thus,

$$\int \frac{(\partial_e f)^2}{(\partial_e f)^2 + \varepsilon} \varphi'(f) d\mu \leq \varepsilon^{-1/2} 2^{-1} (1 + 3d\pi) \|D_e \mu\|_{\text{TV}}.$$

Let $M = \|\varphi'\|_\infty$. Let us estimate the expression

$$M^{-1} \int \varphi'(f) ((\partial_e f)^2 + \varepsilon)^{-1} d\mu.$$

This expression can be estimated from above by

$$\begin{aligned} & \int ((\partial_e f)^2 + \varepsilon)^{-1} d\mu = \int_0^{1/\varepsilon} \mu(((\partial_e f)^2 + \varepsilon)^{-1} > t) dt = \int_0^{1/\varepsilon} \mu((\partial_e f)^2 < 1/t - \varepsilon) dt \\ & = \int_0^\infty (s + \varepsilon)^{-2} \mu((\partial_e f)^2 < s) ds \leq c_1 d \|\partial_e f\|_2^{-1/(d-1)} \int_0^\infty (s + \varepsilon)^{-2} s^{1/(2d-2)} ds \\ & = c_1 d \int_0^\infty (s + 1)^{-2} s^{1/(2d-2)} ds \|\partial_e f\|_2^{-1/(d-1)} \varepsilon^{-1+1/(2d-2)}, \end{aligned}$$

where the Carbery–Wright inequality from Theorem 1.3 was used in the last step. Hence

$$\int \varphi'(f) d\mu \leq c_1(d) \|\partial_e f\|_2^{-1/(d-1)} M \varepsilon^{1/(2d-2)} + c_2(d) \|D_e \mu\|_{\text{TV}} \varepsilon^{-1/2},$$

where

$$c_1(d) = cd \int_0^\infty (s + 1)^{-2} s^{1/(2d-2)} ds, \quad c_2(d) = 2^{-1} (1 + 3d\pi).$$

Set $\varepsilon = M^{-2+2/d}$. Then

$$\int \varphi'(f) d\mu \leq \left(c_1(d) \|\partial_e f\|_2^{-1/(d-1)} + c_2(d) \|D_e \mu\|_{TV} \right) M^{1-1/d}.$$

The lemma is proved. \square

Corollary 4.2. *Let $n, d \in \mathbb{N}$. Then there is a constant $c(d, n)$ depending only on d and n such that, whenever μ is a log-concave measure on \mathbb{R}^n with a density ρ , f is a polynomial of degree d on \mathbb{R}^n , for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ the following inequality holds:*

$$\int \varphi'(f) d\mu \leq c(d, n) \left(\left(\int |\nabla f|^{1/(d-1)} d\mu \right)^{-1} + \max_{|e|=1} \|D_e \mu\|_{TV} \right) \|\varphi'\|_\infty^{1-1/d}.$$

Proof. Multiplying the inequality from Lemma 4.1 by $\int |(\nabla f, e)|^{1/(d-1)} d\mu$ and using the inequality

$$\int |(\nabla f, e)|^{1/(d-1)} d\mu \leq \left(\int |(\nabla f, e)|^2 d\mu \right)^{1/(2d-2)},$$

we get the estimate

$$\int |(\nabla f, e)|^{1/(d-1)} d\mu \int \varphi'(f) d\mu \leq \left(c_1(d) + c_2(d) \|D_e \mu\|_{TV} \int |(\nabla f, e)|^{1/(d-1)} d\mu \right) \|\varphi'\|_\infty^{1-1/d}.$$

Estimating $\|D_e \mu\|_{TV}$ by the maximum on the sphere and integrating with respect to the standard normalized surface measure σ_n on the sphere, we obtain

$$\begin{aligned} & \int_{S^{n-1}} \int |(\nabla f, e)|^{1/(d-1)} d\mu \sigma_n(de) \int \varphi'(f) d\mu \\ & \leq \left(c_1(d) + c_2(d) \max_{|e|=1} \|D_e \mu\|_{TV} \int_{S^{n-1}} \int |(\nabla f, e)|^{1/(d-1)} d\mu \sigma_n(de) \right) \|\varphi'\|_\infty^{1-1/d}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{S^{n-1}} \int |(\nabla f, e)|^{1/(d-1)} d\mu \sigma_n(de) = \int \int_{S^{n-1}} |(\nabla f, e)|^{1/(d-1)} \sigma_n(de) d\mu \\ & = \int |\nabla f|^{1/(d-1)} \int_{S^{n-1}} |(e, e_1)|^{1/(d-1)} \sigma_n(de) d\mu = C(n, d) \int |\nabla f|^{1/(d-1)} d\mu, \end{aligned}$$

where

$$C(n, d) = \int_{S^{n-1}} |(e, e_1)|^{1/(d-1)} \sigma_n(de).$$

Thus, the assertion of the corollary is true with the constant

$$c(d, n) = \max \left\{ \frac{c_1(d)}{C(n, d)}, c_2(d) \right\}.$$

The corollary is proved. \square

Corollary 4.3. *Let $d, n \in \mathbb{N}$. Then, there is a constant $C(d, n)$ depending only on d and n such that, whenever μ is an isotropic log-concave measure on \mathbb{R}^n with a density ρ and the unit isotropic constant and f is a polynomial of degree d on \mathbb{R}^n , for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has*

$$\int \varphi'(f) d\mu \leq C(d, n) \left(\left(\int |f - \mathbb{E}f|^{1/(d-1)} d\mu \right)^{-1} + 1 \right) \|\varphi'\|_\infty^{1-1/d}.$$

Proof. By Theorem 1.4 for the log-concave measure μ we have

$$\int (f - \mathbb{E}f)^2 d\mu \leq R \int |x - x_0|^2 d\mu \int |\nabla f|^2 d\mu,$$

where R is an absolute constant, $x_0 = \int x d\mu$. Note that

$$\int |x - x_0|^2 d\mu = \int |x|^2 d\mu = n.$$

By Theorem 3.4 we have $\|D_e \mu\| \leq C(n)$. Using the estimates from Theorem 1.2 and Corollary 4.2, we obtain the desired inequality. \square

Theorem 4.4. *Let $d \geq 2$. Then, there is a constant $C(d)$ depending only on d such that, whenever μ is a log-concave measure on \mathbb{R}^n , f is a polynomial of degree d on \mathbb{R}^n , for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has*

$$\sigma_f^{1/d} \int \varphi'(f) d\mu \leq C(d) \|\varphi'\|_\infty^{1-1/d}.$$

Proof. First, we prove the estimate

$$(4.1) \quad \int \varphi'(f) d\mu \leq C_1(d) \left(\left(\int |f - \mathbb{E}f|^{1/(d-1)} d\mu \right)^{-1} + 1 \right) \|\varphi'\|_\infty^{1-1/d}.$$

Note that it is sufficient to prove this estimate only for log-concave measures with compact support. Let us fix a log-concave measure μ with a convex compact support K and a polynomial f of degree d . Without loss of generality we can assume that $\mathbb{E}f := \int f d\mu = 0$. Let $k := \int |f|^{1/(d-1)} d\mu$. Let P_f be the set of all log-concave measures ν with support in K such that

$$\int f d\nu \geq 0, \quad \int -f d\nu \geq 0, \quad \int |f|^{1/(d-1)} d\nu \geq k.$$

Note that $\mu \in P_f$. Consider the function

$$\Phi_f(\nu) := \int \varphi'(f) d\nu - C_1(d) (k^{-1} + 1) \|\varphi'\|_\infty^{1-1/d},$$

where $C_1(d) = \max\{C(d, n), n = 1, 2, 3\}$, $C(d, n)$ is the constant from the previous corollary. Now we prove that $\Phi_f(\nu) \leq 0$ for every polynomial f of degree d and every log-concave measure $\nu \in P_f$. Due to Theorem 1.5 (with $p = 3$, $f_1 = f$, $f_2 = -f$, $f_3 = |f|^{1/(d-1)}$) it is sufficient to prove this inequality for every polynomial f of degree d and every log-concave measure $\nu \in P_f$ such that the smallest affine subspace containing the support of ν is of dimension not greater than 3. For every log-concave measure ν with a density, there is a non-degenerate linear transformation T such that the measure $\nu \circ T^{-1}$ is isotropic with the unit isotropic constant. Since a polynomial of degree d composed with a linear transformation is a polynomial of degree d , it is sufficient to prove our inequality for every isotropic log-concave measure ν with the unit isotropic constant in dimensions at most 3 and every polynomial f of degree d . By Corollary 4.3 one has $\Phi_f(\nu) \leq 0$ for such measures and polynomials. Thus, $\Phi_f(\nu) \leq 0$ for every polynomial f of degree d and every log-concave measure $\nu \in P_f$, and since $\mu \in P_f$ and $k = \int |f|^{1/(d-1)} d\mu$, we have (4.1). By Theorem 1.2

$$\int |f - \mathbb{E}f|^{1/(d-1)} d\mu \geq (4c(d-1))^{-1} \left(\int |f - \mathbb{E}f|^2 d\mu \right)^{1/(2d-2)}.$$

Let f be a polynomial of degree d on \mathbb{R}^n . Then by inequality (4.1) we have

$$\int \varphi'(f \sigma_f^{-1}) d\mu \leq C_1(d) (4c(d-1) + 1) \|\varphi'\|_\infty^{1-1/d}.$$

Let $\psi(t) = \varphi(t\sigma_f^{-1})$, $C(d) = C_1(d)(4c(d-1) + 1)$. Then

$$\int \psi'(f) d\mu = \sigma_f^{-1} \int \varphi'(f\sigma_f^{-1}) d\mu \leq C(d)\sigma_f^{-1} \|\varphi'\|_\infty^{1-1/d} = C(d)\sigma_f^{-1/d} \|\psi'\|_\infty^{1-1/d}.$$

Thus, the theorem is now proved. \square

5. PROPERTIES OF POLYNOMIAL IMAGES OF LOG-CONCAVE MEASURES

Here we use an approximation argument, Theorem 4.4, and the results of Section 2 to obtain the main results of this work: Corollaries 5.2, 5.3, 5.4, and 5.5. In the formulations below by non-constant functions we mean functions that do not coincide with constants almost everywhere.

Theorem 5.1. *Let $d \in \mathbb{N}$. Then, there is a constant $C(d)$ depending only on d such that, whenever μ is a Radon log-concave measure on a locally convex space E and $f \in \mathcal{P}^d(\mu)$, for every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$ one has*

$$(5.1) \quad \sigma_f^{1/d} \int \varphi'(f) d\mu \leq C(d) \|\varphi'\|_\infty^{1-1/d}.$$

Proof. Let $f_n = f_n(\ell_{1,n}, \dots, \ell_{k_n,n})$ be a sequence of polynomials of degree d in finitely many variables such that its limit in $L^2(\mu)$ is f . The polynomials f_n satisfy inequality (5.1) by Theorem 4.4. Since

$$\begin{aligned} \int \varphi'(f_n) d\mu &\rightarrow \int \varphi'(f) d\mu, \\ \int (f_n - \mathbb{E}f_n)^2 d\mu &\rightarrow \int (f - \mathbb{E}f)^2 d\mu, \end{aligned}$$

this inequality is also valid for the function f . \square

Corollary 5.2. *Let μ be a Radon log-concave measure on a locally convex space E . Let $C(d)$ be the same constant as in Theorem 5.1 and let $f \in \mathcal{P}^d(\mu)$. Then*

$$\begin{aligned} \sigma_f^{1/d} \|(\mu \circ f^{-1})_h - \mu \circ f^{-1}\|_{TV} &\leq 2^{1-1/d} C(d) |h|^{1/d}, \\ \sigma_f^{1/d} \mu \circ f^{-1}(A) &\leq C(d) \lambda(A)^{1/d}, \end{aligned}$$

where λ is the standard Lebesgue measure on the real line, A is a Borel set on the real line. In particular, the measure $\mu \circ f^{-1}$ is absolutely continuous.

Proof. We apply Lemma 2.1 and Lemma 2.4. \square

Denote by ρ_f the density of the measure $\mu \circ f^{-1}$, which exists by the previous corollary. The next corollary follows from Lemma 2.4.

Corollary 5.3. *Let μ be a Radon log-concave measure on a locally convex space E and let $f \in \mathcal{P}^d(\mu)$. Then $\rho_f \in L^p(\mathbb{R})$ whenever $1 < p < \frac{d}{d-1}$, and for the L^p -norm of ρ_f the following inequality holds:*

$$\sigma_f^{1-1/p} \|\rho_f\|_{L^p(\mathbb{R})} \leq C_1(d, p),$$

where

$$C_1(d, p) = \left(p(p-1)^{-1} + p \left(\frac{d}{d-1} - p \right)^{-1} \right)^{1/p} C(d)^{d(1-1/p)}.$$

Our next corollary generalizes some results from [25, 26] and [11] to the case of arbitrary log-concave measures in place of Gaussian measures, and even in the Gaussian case this estimate provides a better rate of convergence as compared to analogous estimates from [25] and [11] in the one-dimensional case. It follows directly from Lemma 2.3 and Theorem 5.1.

Corollary 5.4. *Let μ be a Radon log-concave measure on a locally convex space E . Then for every pair of non-constant functions $f, g \in \mathcal{P}^d(\mu)$ one has*

$$\|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{TV}} \leq C_d(\sigma_f, \sigma_g) \|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{FM}}^{1/(1+d)},$$

where

$$C_d(\sigma_f, \sigma_g) = 1 + 2C(d)(\sigma_f^{-1/d} + \sigma_g^{-1/d})(2\pi)^{-1/2} \int e^{-t^2/2} |t|^{1/d} dt.$$

Combining Remark 2.5 with the previous lemma, we obtain the following result.

Corollary 5.5. *Let μ be a Radon log-concave measure on a locally convex space E . Then for every pair of non-constant functions $f, g \in \mathcal{P}^d(\mu)$ one has*

$$\begin{aligned} \|\rho_f - \rho_g\|_{L^p(\mathbb{R})} &\leq C_1(d, p)(\sigma_f^{-1/d} + \sigma_g^{-1/d})^{d(1-1/p)} \|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{TV}}^{1-d(1-1/p)} \\ &\leq C_1(d, p)(\sigma_f^{-1/d} + \sigma_g^{-1/d})^{d(1-1/p)} C_d(\sigma_f, \sigma_g)^{1-d(1-1/p)} \|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\text{FM}}^{\frac{1}{1+d}(1-d(1-1/p))}, \end{aligned}$$

where ρ_f, ρ_g are densities of the measures $\mu \circ f^{-1}, \mu \circ g^{-1}$, respectively, and $1 < p < \frac{d}{d-1}$.

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